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A common fixed point theorem for rational inequality in Hilbert space

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ABSTRACT : The object of this paper is to obtain a common unique fixed-point theorem for four continuous mappings defined on a non-empty closed subset of a Hilbert space.

Keywords : Hilbert space, common fixed point, rational inequality.

Mathematics Subject Classification: 47H10, 54H25

I. INTRODUCTION

The Study of properties and applications of fixed points of various types of contractive mapping in Hilbert and Banach spaces were obtained among others by Browder [1], Browder and Petryshyn [2,3], Hicks and Huffman [5], Huffman [6], Koparde and Waghmode [7]. In this paper we present a common fixed point theorem for rational inequality involving self mappings. For the purpose of obtaining the fixed point of the four continuous mappings. We have constructed a sequence and have shown its convergence to the fixed point.

II. MAIN RESULT

Theorem 1: *Let E, F, T and S be for continuous self mappings of a closed subset C of a Hilbert Space H satisfying $ES = SE$, $FT = TF$, $E(H) \subset T(H)$ and $F(H) \subseteq S(H)$*

...(2.1)

$$\|Ex-Fy\|^2 \le a_1 \frac{\|Sx-Ex\|^2[\|Ty-Fy\|^2+\|Ex-Ty\|^2]}{\|Sx-Ty\|^2+\|Ex-Ty\|^2} + a_2 \frac{\|Ex-Ty\|^2\|Sx-Fy\|^2[\|Sx-Ex\|^2+\|Ty-Fy\|^2]}{\|Sx-Ty\|^2+\|Ex-Ty\|^2} + a_3 \frac{\|Ty-Fy\|^2\|Sx-Ex\|^2}{\|Sx-Ty\|^2} + a_4[\|Ex-Ty\|^2+\|Sx-Fy\|^2] + a_5[\|Sx-Ex\|^2+\|Ty-Fy\|^2]+a_6\|Sx-Ty\|^2 \quad ...(2.2)$$

for all $x, y \in C$ with $Sx \neq Ty$, and

$$\|Sx-Ty\|^2+\|Ex-Ty\|^2 \neq 0,$$

$a_1, a_2, a_3, a_4, a_5, a_6 \ge 0$ and $(a_1 + a_3 + 2a_4 + a_5 + a_6) < \frac{1}{2}$ ...(2.3)

Then E, F, T, and S has a unique common fixed point

Proof : *Let $x_0 \in C$* by (1) there exists a point $x_1 \in C$ such that $Tx_1 = Ex_0$ and for this point x_1 we can choose a point $x_2 \in C$ such that $Fx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in C such that

$$y_{2n} = Tx_{2n+1} = Ex_{2n} \text{ and}$$

$$y_{2n+1} = Sx_{2n+2} = Fx_{2n+1},\ n = 0, 1, 2, 3, 4, ... \quad ...(2.4)$$

From (2.2) and (2.4) we have

$$\|y_{2n}-y_{2n+1}\|^2 = \|Ex_{2n}-Fx_{2n+1}\|^2$$

$$\le a_1 \frac{\|Sx_{2n}-Ex_{2n}\|^2[\|Tx_{2n+1}-Fx_{2n+1}\|^2+\|Ex_{2n}-Tx_{2n+1}\|^2]}{\|Sx_{2n}-Tx_{2n+1}\|^2+\|Ex_{2n}-Tx_{2n+1}\|^2}$$

$$+ a_2 \frac{\|Ex_{2n}-Tx_{2n+1}\|^2\|Sx_{2n+1}-Fx_{2n+1}\|^2[\|Sx_{2n}-Ex_{2n}\|^2+\|Tx_{2n+1}-Fx_{2n+1}\|^2]}{\|Sx_{2n}-Tx_{2n+1}\|^2+\|Ex_{2n}-Tx_{2n+1}\|^2}$$

$$+ a_3 \frac{\|Tx_{2n+1}-Fx_{2n+1}\|^2\|Sx_{2n}-Ex_{2n}\|^2}{\|Sx_{2n}-Tx_{2n+1}\|^2}$$

$$+ a_4\|Ex_{2n}-Tx_{2n+1}\|^2+\|Sx_{2n}-Fx_{2n+1}\|^2$$

$$+ a_5[\|Sx_{2n}-Ex_{2n}\|^2+\|Tx_{2n+1}-Fx_{2n+1}\|^2]+a_6\|Sx_{2n}-Tx_{2n+1}\|^2$$

$$= a_1 \frac{\|y_{2n-1}-y_{2n}\|^2[\|y_{2n}-y_{2n+1}\|^2+\|y_{2n}-y_{2n}\|^2]}{\|y_{2n-1}-y_{2n}\|^2+\|y_{2n}-y_{2n}\|^2}$$

$$+ a_2 \frac{\|y_{2n}-y_{2n}\|^2\|y_{2n-1}-y_{2n+1}\|^2[\|y_{2n-1}-y_{2n}\|^2+\|y_{2n}-y_{2n+1}\|^2]}{\|y_{2n-1}-y_{2n}\|^2+\|y_{2n}-y_{2n}\|^2}$$

$$+ a_3 \frac{\|y_{2n}-y_{2n+1}\|^2\|y_{2n-1}-y_{2n}\|^2}{\|y_{2n-1}-y_{2n}\|^2}$$

$$+ a_4[\|y_{2n}-y_{2n}\|^2+\|y_{2n-1}-y_{2n+1}\|^2]$$

$$+ a_5[\|y_{2n-1}-y_{2n}\|^2+\|y_{2n}-y_{2n+1}\|^2]+a_6\|y_{2n-1}-y_{2n}\|^2$$

$$\Rightarrow \|y_{2n}-y_{2n+1}\|^2 \le (a_1+a_3+2a_4+a_5)\|y_{2n}-y_{2n+1}\|^2 + (2a_4+a_5+a_6)\|y_{2n-1}-y_{2n}\|^2$$

$$\Rightarrow \|y_{2n}-y_{2n+1}\|^2 \le \left[\frac{2a_4+a_5+a_6}{1-(a_1+a_3+2a_4+a_5)}\right]\|y_{2n-1}-y_{2n}\|^2$$

$$\Rightarrow \|y_{2n} - y_{2n+1}\| \le \left[\frac{(a_5 + a_6 + 2a_4)}{1-(a_1 + a_3 + 2a_4 + a_5)}\right]^{1/2} \|y_{2n-1} - y_{2n}\|$$

$$\Rightarrow \|y_{2n} - y_{2n+1}\| \le k\|y_{2n-1} - y_{2n}\|$$

Where $k = \left[\frac{2a_4 + a_5 + a_6}{1-(a_1 + a_3 + 2a_4 + a_5)}\right]^{1/2} < 1$ (by 2.3)

and

$$\|y_{2n} - y_{2n-1}\|^2 = \|Ex_{2n} - Fx_{2n-1}\|^2$$

$$\le a_1 \frac{\|Sx_{2n} - Ex_{2n}\|^2 [\|Tx_{2n-1} - Fx_{2n-1}\|^2 + \|Ex_{2n} - Tx_{2n-1}\|^2]}{\|Sx_{2n} - Tx_{2n-1}\|^2 + \|Ex_{2n} - Tx_{2n-1}\|^2}$$

$$+ a_2 \frac{\|Ex_{2n} - Tx_{2n-1}\|^2 \|Sx_{2n} - Fx_{2n-1}\|^2 [\|Sx_{2n} - Ex_{2n}\|^2 + \|Tx_{2n-1} - Fx_{2n-1}\|^2]}{\|Sx_{2n} - Tx_{2n-1}\|^2 + \|Ex_{2n} - Tx_{2n-1}\|^2}$$

$$+ a_3 \frac{\|Tx_{2n-1} - Fx_{2n-1}\|^2 \|Sx_{2n} - Ex_{2n}\|^2}{\|Sx_{2n} - Tx_{2n-1}\|^2}$$

$$+ a_4[\|Ex_{2n} - Tx_{2n-1}\|^2 + \|Sx_{2n} - Fx_{2n-1}\|^2]$$

$$+ a_5[\|Sx_{2n} - Ex_{2n}\|^2 + \|Tx_{2n-1} - Fx_{2n-1}\|^2]$$

$$+ a_6 \|Sx_{2n} - Tx_{2n-1}\|^2$$

$$= a_1 \frac{\|y_{2n-1} - y_{2n}\|^2 [\|y_{2n-2} - y_{2n-1}\|^2 + \|y_{2n} - y_{2n-2}\|^2]}{\|y_{2n-1} - y_{2n-2}\|^2 + \|y_{2n} - y_{2n-2}\|^2}$$

$$+ a_2 \frac{\|y_{2n} - y_{2n-2}\|^2 \|y_{2n-1} - y_{2n-1}\|^2 [\|y_{2n-1} - y_{2n}\|^2 + \|y_{2n-2} - y_{2n-1}\|^2]}{\|y_{2n-1} - y_{2n-2}\|^2 + \|y_{2n} - y_{2n-2}\|^2}$$

$$+ a_3 \frac{\|y_{2n-2} - y_{2n-1}\|^2 \|y_{2n-1} - y_{2n}\|^2]}{\|y_{2n-1} - y_{2n-2}\|^2}$$

$$+ a_4\left[\|y_{2n} - y_{2n-2}\|^2 + \|y_{2n-1} - y_{2n-1}\|^2\right]$$

$$+ a_5\left[\|y_{2n-1} - y_{2n}\|^2 + \|y_{2n-2} - y_{2n-1}\|^2\right]$$

$$+ a_6 \|y_{2n-1} - y_{2n-2}\|^2$$

$$\Rightarrow \|y_{2n} - y_{2n-1}\|^2 \le (a_1 + a_3 + 2a_4 + a_5)\|y_{2n-1} - y_{2n}\|^2$$

$$+ (2a_4 + a_5 + a_6)\|y_{2n-1} - y_{2n-2}\|^2$$

$$\Rightarrow [1-(a_1 + a_3 + 2a_4 + a_5)]\|y_{2n} - y_{2n-1}\|^2$$

$$\le (2a_4 + a_5 + a_6)\|y_{2n-1} - y_{2n-2}\|^2$$

$$\Rightarrow \|y_{2n} - y_{2n-1}\|^2 \le \left[\frac{2a_4 + a_5 + a_6}{1-(a_1 + a_3 + 2a_4 + a_5)}\right]\|y_{2n-1} - y_{2n-2}\|^2$$

$$\Rightarrow \|y_{2n} - y_{2n-1}\| \le \left[\frac{2a_4+a_5+a_6}{1-(a_1+a_3+2a_4+a_5)}\right]^{1/2} \|y_{2n-1} - y_{2n-2}\|$$

$$\Rightarrow \|y_{2n} - y_{2n-1}\| \le k\|y_{2n-1} - y_{2n-2}\| \quad ...(2.6)$$

Where $k = \left[\frac{2a_4 + a_5 + a_6}{1-(a_1 + a_3 + 2a_4 + a_5)}\right]^{1/2} < \frac{1}{2}$ (by 2.3)

Equation (2.5) and (2.6) jointly imply that for n = 1, 2, 3

$$\|y_n - y_{n+1}\| \le k\|y_{n-1} - y_n\|$$

$$\Rightarrow \|y_n - y_{n+1}\| \le k^n\|y_0 - y_1\|$$

Now, we shall prove that $\{y_n\}$ is a Cauchy sequence for this every positive integer p we have

$$\|y_n - y_{n+p}\|$$

$$= \|y_n - y_{n+1} + y_{n+1} - y_{n+2} + y_{n+2} + ... + y_{n+p-1} - y_{n+p}\|$$

$$\le \|y_n - y_{n+1}\| + \|y_{n+1} - y_{n+2}\| + ... + \|y_{n+p-1} - y_{n+p}\|$$

$$\le [k^n + k^{n+1} + ... + k^{n+p-1}]\|y_0 - y_1\|$$

$$= k^n[1 + k + k^2 + ... + k^{p-1}]\|y_0 - y_1\|$$

$$< \frac{k^n}{(1-k)}\|y_0 - y_1\|$$

$$\Rightarrow \|y_n - y_{n+p}\| \to 0 \text{ as } n \to \infty$$

Therefore $\{y_n\}$ is a Cauchy sequence in C. Then as C is closed subset of H So $\{y_n\}_{n \in N}$ converges to some $u \in C$. So consequently the subsequences $\{Tx_{2n+1}\}$, $\{Ex_{2n+1}\}$ and $\{Sx_{2n+2}\}$ of $\{y_n\}$ also converges to the same point u. Now from *(2.1)* since E, F, T and S are continuous such that

$$E(S(x_n)) \to Eu$$
$$S(E(x_n)) \to Su$$
$$F(T(x_n)) \to Fu$$
and $$T(F(x_n)) \to Tu \quad (\text{as } n \to \infty)$$

$Eu = Su$, $Fu = Tu$ from (2.1)

Existence of Fixed point : Consider

$$\|Eu - u\|^2 = \|Eu - y_{2n+1} + y_{2n+1} - u\|^2$$

$$\le 2\|Eu - y_{2n+1}\|^2 + 2\|y_{2n+1} - u\|^2$$

$$= 2\|Eu - Fx_{2n+1}\|^2 + 2\|y_{2n+1} - u\|^2$$

$$\le 2a_1 \frac{\|Su - Eu\|^2 [\|Tx_{2n+1} - Fx_{2n+1}\|^2 + \|Eu - Tx_{2n+1}\|^2]}{\|Su - Tx_{2n+1}\|^2 + \|Eu - Tx_{2n+1}\|^2}$$

$$+ 2a_2 \frac{\|Eu - Tx_{2n+1}\|^2 \|Su - Fx_{2n+1}\|^2 [\|Su - Eu\|^2 + \|Tx_{2n+1} - Fx_{2n+1}\|^2]}{\|Su - Tx_{2n+1}\|^2 + \|Eu - Tx_{2n+1}\|^2}$$

$$+ 2a_3 \frac{\|Tx_{2n+1} - Fx_{2n+1}\|^2 \|Su - Eu\|^2}{\|Su - Tx_{2n+1}\|^2}$$

$$+ 2a_4[\|Eu - Tx_{2n+1}\|^2 + \|Su - Fx_{2n+1}\|^2]$$

$$+2a_5[\|Su - Eu\|^2 + \|Tx_{2n+1} - Fx_{2n+1}\|^2]$$

$$+2a_6 \|Su - Tx_{2n+1}\|^2 + \|y_{2n+1} - u\|^2$$

$$\leq 2a_1 \frac{\|Su-Eu\|^2 [\|u-u\|^2 + \|Eu-u\|^2]}{\|Su-u\|^2 + \|Eu-u\|^2}$$

$$+ 2a_2 \frac{\|Eu-u\|^2 \|Su-u\|^2 [\|Su-Eu\|^2 + \|u-u\|^2]}{\|Su-u\|^2 + \|Eu-u\|^2}$$

$$+ 2a_3 \frac{\|u-u\|^2 + \|Su-Eu\|^2}{\|Su-u\|^2} + 2a_4[\|Eu-u\|^2 + \|Su-u\|^2]$$

$$+ 2a_5[\|Su-Eu\|^2 + \|u-u\|^2] + 2a_6 \|Su-u\|^2 + 2\|u-u\|^2$$

(as $n \to \infty$)

Therefore

$$\|Eu-u\|^2 \leq (4a_4 + 2a_6)\|Eu-u\|^2 \quad [\text{as } Su = Eu]$$

$$\Rightarrow (1 - 4a_4 - 2a_6)\|Eu-u\|^2 \leq 0$$

$$\Rightarrow \|Eu-u\|^2 = 0 \qquad [as\ 4a_4 + 2a_2 < 1]$$

$$\Rightarrow Eu = u$$

This gives $Eu = u = Su$ *i.e.* u is a fixed point of E and S. Similarly we have $Fu = u = Tu$ So u is a common fixed point of E, F, S and T.

Uniqueness : *Let v be another fixed point of E, F, T and S then*

$$\|u-v\|^2 = \|Eu-Fv\|^2$$

$$\leq a_1 \frac{\|Su-Eu\|^2 [\|Tv-Fv\|^2 + \|Eu-Tv\|^2]}{\|Su-Tv\|^2 + \|Eu-Tv\|^2}$$

$$+ a_2 \frac{\|Eu-Tv\|^2 \|Su-Fv\|^2 [\|Su-Eu\|^2 + \|Tv-Fv\|^2]}{\|Su-Tv\|^2 + \|Eu-Tv\|^2}$$

$$+ a_3 \frac{\|Tv-Fv\|^2 \|Su-Eu\|^2}{\|Su-Tv\|^2}$$

$$+ a_4[\|Eu-Tv\|^2 + \|Su-Fv\|^2]$$

$$+ a_5[\|Su-Eu\|^2 + \|Tv-Fv\|^2] + a_6 \|Su-Tv\|^2$$

$$\|u-v\|^2 \leq (2a_4 + a_6)\|u-v\|^2$$

$$\Rightarrow (1 - 2a_4 - a_6)\|u-v\|^2 \leq 0$$

$$\Rightarrow \|u-v\|^2 = 0 \qquad [as\ 2a_4 + a_6 < 1]$$

$$\Rightarrow u = v$$

Hence u is a unique common fixed point of E, F, S and T. This completes the proof.

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